

2023年秋试卷答案解析

2024 年 1 月 10 日

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1. $\sigma^2 \frac{1+2\phi_1\theta_1+\theta_1^2}{1-\phi_1^2}, \sigma^2(\theta_1+\phi_1)\phi_1^{h-1} \frac{1+\phi_1\theta_1}{1-\phi_1^2}$

2. $\frac{\theta_1}{1+\theta_1^2}, \frac{\theta_1\theta_{12}}{(1+\theta_1^2)(1+\theta_{12}^2)}$

3. B

4. $-0.9 < a < 1$ 且 $a \neq -0.2$

5. 0, $\sigma^2 \cos(\lambda_0 k)$, 是,

$$\begin{cases} \frac{\sigma^2}{2} [I_{[-\lambda_0, \pi]}(\lambda) + I_{[\lambda_0, \pi]}(\lambda)] & \text{if } \lambda_0 \neq \pi \\ \sigma^2 I_{\{\pi\}}(\lambda) & \text{if } \lambda_0 = \pi \end{cases}$$

6. $\frac{t}{\sqrt{t(t+k)}}, <$

7. $(1 + \theta_1^2 + \cdots + \theta_{l-1}^2)\sigma^2$

8. $(1-p)^2 \gamma_k$

二

(1). 由谱密度公式 $f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|A(e^{i\lambda})|^2}$, $A(z) = 1 - 0.6z^3$, 得到

$$f_X(\lambda) = \frac{1}{2\pi} \frac{1}{|1 - 0.6e^{3i\lambda}|^2} = \frac{1}{2\pi} \frac{1}{1.36 - 1.2\cos(3\lambda)}$$

(2). 由线性滤波结论: $Y_t = \sum_{j=-\infty}^{\infty} h_j X_{t-j}$, $H(z) = \sum_{j=-\infty}^{\infty} h_j z^j$, $f_Y(\lambda) = |H(e^{-i\lambda})|^2 f_X(\lambda)$
代入本题有 $H(e^{-i\lambda}) = \sum_{j=-\infty}^{\infty} \gamma_j e^{-ij\lambda} = 2\pi f_X(\lambda)$, 从而

$$f_Y(\lambda) = 4\pi^2 f_X^3(\lambda) = \frac{1}{2\pi} \frac{1}{(1.36 - 1.2\cos(3\lambda))^3}.$$

三

(1). 由题意, $\eta_t = \varepsilon_t^2 - \sigma_t^2$, 代入 $\sigma_t^2 = w + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$ 有

$$\varepsilon_t^2 = w + \alpha\varepsilon_{t-1}^2 + \beta(\varepsilon_{t-1}^2 - \eta_{t-1}) + \eta_t = w + (\alpha + \beta)\varepsilon_{t-1}^2 + \eta_t - \beta\eta_{t-1}.$$

而中心化后根据题目条件有 $A(z) = 1 - (\alpha + \beta)z$ 根在单位圆外, $B(z) = 1 - \beta z$ 根不在单位圆内, 从而 $\{\varepsilon_t^2\}$ 满足非中心的 $ARMA(1, 1)$.

- (2). $\sigma_\eta^2 = E[\eta_t^2] = E\{E[(\epsilon_t^2 - \sigma_t^2)^2|F_{t-1}]\} = E\{E[\epsilon_t^4 + \sigma_t^4 - 2\sigma_t^2\epsilon_t^2|F_{t-1}]\}$, F_{t-1} 表示截止到 $t-1$ 时刻 ϵ_{t-j} 和 σ_{t-j} 包含的信息.

其中

$$\begin{aligned} E\{E[\sigma_t^4|F_{t-1}]\} &= E[\sigma_t^4] = \frac{1}{3}E[\sigma_t^4 E(\nu_t^4)] (X \sim N(0, 1), EX^4 = 3) \\ &= \frac{1}{3}E\{E[\sigma_t^4 \nu_t^4|F_{t-1}]\} = \frac{1}{3}E\{E[\epsilon_t^4|F_{t-1}]\} = \frac{1}{3}E[\epsilon_t^4], \\ E\{E[\sigma_t^2 \epsilon_t^2|F_{t-1}]\} &= E[\sigma_t^4 E(\nu_t^2)] = E[\sigma_t^4] = \frac{1}{3}E\{E[\epsilon_t^4|F_{t-1}]\} = \frac{1}{3}E[\epsilon_t^4], \end{aligned}$$

则有 $\sigma_\eta^2 = 2/3E[\epsilon_t^4]$, 下求 $E[\epsilon_t^4]$.

代入模型有

$$\begin{aligned} E[\epsilon_t^4] &= 3E[(\omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2)^2] \\ &= 3(\omega^2 + \alpha^2 E[\epsilon_{t-1}^4] + \beta^2 E[\sigma_{t-1}^4] + 2\alpha\omega E[\epsilon_{t-1}^2] + 2\omega\beta E[\sigma_{t-1}^2] + 2\alpha\beta E[\epsilon_{t-1}^2 \sigma_{t-1}^2]) \\ &= 3\left(\omega^2 + \alpha^2 E[\epsilon_{t-1}^4] + \frac{1}{3}\beta^2 E\{E[\epsilon_{t-1}^4|F_{t-2}]\} + 2\omega(\alpha + \beta)E[\epsilon_{t-1}^2] + \frac{2}{3}\alpha\beta E\{E[\epsilon_{t-1}^4|F_{t-2}]\}\right). \end{aligned}$$

利用 ϵ_t 序列存在严平稳解, 令 $E[\epsilon_t^4] = E[\epsilon_{t-1}^4]$, 整理后得到

$$\begin{aligned} [1 - (3\alpha^2 + \beta^2 + 2\alpha\beta)]E[\epsilon_t^4] &= 3\frac{\omega^2(1 + \alpha + \beta)}{1 - \alpha - \beta} \\ \sigma_\eta^2 &= \frac{2\phi_0}{1 - \phi_1}, \quad \phi_0 = \frac{\omega^2(1 + \alpha + \beta)}{1 - \alpha - \beta}, \quad \phi_1 = 3\alpha^2 + \beta^2 + 2\alpha\beta. \end{aligned}$$

四

- (1). 由题意、YW方程和书P149性质6, 有

$$\begin{aligned} \hat{X}_{n+2} &= L(X_{n+2}|X_{n+1}, \dots, X_1) \\ &= L(X_{n+2}|X_{n+1}, \dots, X_2, X_1 - \tilde{X}_1) = L(X_{n+2}|X_{n+1}, \dots, X_2) + L(X_{n+2}|X_1 - \tilde{X}_1) \\ &= a_{n1}X_{n+1} + \dots + a_{nn}X_2 + \lambda(X_1 - \tilde{X}_1). \end{aligned}$$

下面确定 λ . 由 $X_{n+2} = X_{n+2} - \hat{X}_{n+2} + \hat{X}_{n+2}$, $X_{n+2} - \hat{X}_{n+2} \perp \hat{X}_{n+2}$, 两边乘 X_1 取期望得到

$$\gamma_{n+1} = a_{n1}\gamma_n + \dots + a_{nn}\gamma_1 + \lambda E[(X_1 - \tilde{X}_1)^2].$$

下面则求 $E[(X_1 - \tilde{X}_1)^2]$.

思路一: 根据YW方程预测有 $L(X_1|X_{n+1}, \dots, X_2) = a_{n1}X_2 + \dots + a_{nn}X_{n+1}$ (a_{n1} 对应的是最邻近的观测值), 则根据YW方程计算预测误差可以得到

$$\begin{aligned} E[(X_1 - \tilde{X}_1)^2] &= \gamma_0 - \mathbf{a}^T \tilde{\gamma}_n, \quad \tilde{\gamma}_n = (\gamma_1, \dots, \gamma_n)' \\ &= \gamma_0 - a_{n1}\gamma_1 - \dots - a_{nn}\gamma_n \end{aligned}$$

思路二: 设 $\tilde{X}_1 = \tilde{\mathbf{a}}^T \tilde{\mathbf{X}}$, $\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_n)'$, $\tilde{\mathbf{X}} = (X_{n+1}, \dots, X_2)'$.

由预测方程有 $\tilde{\mathbf{\Gamma}}\tilde{\mathbf{a}} = E[\tilde{\mathbf{X}}X_1]$, $\tilde{\mathbf{\Gamma}} = E[\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T]$. 即有

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{n-2} \\ \vdots & \vdots & & \vdots \\ \gamma_{n-1} & \gamma_{n-2} & \cdots & \gamma_0 \end{pmatrix} \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \vdots \\ \tilde{a}_n \end{pmatrix} = \begin{pmatrix} \gamma_n \\ \vdots \\ \vdots \\ \gamma_1 \end{pmatrix}$$

而YW方程得到

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{n-2} \\ \vdots & \vdots & & \vdots \\ \gamma_{n-1} & \gamma_{n-2} & \cdots & \gamma_0 \end{pmatrix} \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nn} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

对应系数得到 $\tilde{a}_j = a_{n,n+1-j}$, 从而有 $\tilde{X}_1 = a_{nn}X_{n+1} + \dots + a_{n1}X_2$. 同样是YW方法计算预测误差, 结果同思路一.

最后综合得到

$$\lambda = \frac{\gamma_{n+1} - a_{n,1}\gamma_n - a_{n,2}\gamma_{n-1} - \cdots - a_{n,n}\gamma_1}{\gamma_0 - a_{n,1}\gamma_1 - a_{n,2}\gamma_2 - \cdots - a_{n,n}\gamma_n}.$$

(2). 代入(1)所得结果:

$$\begin{aligned} \hat{X}_{n+2} &= a_{n1}X_{n+1} + \cdots + a_{nn}X_2 + \lambda(X_1 - a_{nn}X_{n+1} - \cdots - a_{n1}X_2) \\ &= (a_{n1} - \lambda a_{nn})X_{n+1} + \cdots + (a_{nn} - \lambda a_{n1})X_2 + \lambda X_1 \\ &= a_{n+1,n+1}X_{n+1} + \cdots + a_{n+1,n}X_2 + a_{n+1,n+1}X_1 \end{aligned}$$

对应系数后得到 $a_{n+1,n+1} = \lambda$, $a_{n+1,j} = a_{n,j} - \lambda a_{n,n+1-j}$,

$$\begin{aligned} \sigma_{n+1}^2 &= \gamma_0 - a_{n+1,1}\gamma_1 - \cdots - a_{n+1,n}\gamma_n - a_{n+1,n+1}\gamma_{n+1} \\ &= \sigma_n^2 + \lambda(a_{n,n}\gamma_1 + \cdots + a_{n,1}\gamma_n - \gamma_{n+1}) \\ &= \sigma_n^2(1 - \lambda^2). \end{aligned}$$

五

(1). 由 $(1 - 1/25B^2)X_t = \epsilon_t$ 和 $1/(1-x) = \sum_{k=0}^{\infty} x^k$ 得到

$$X_t = \sum_{k=0}^{\infty} \left(\frac{1}{25}B^2\right)^k \epsilon_t = \sum_{k=0}^{\infty} \left(\frac{1}{25}\right)^k \epsilon_{t-2k}.$$

(2). 最佳线性预测 $\hat{X}_t(l), l = 1, 2, \dots$ 由下式确定:

$$\begin{cases} \hat{X}_t(l) = \frac{1}{25}\hat{X}_t(l-2), & l = 1, 2, \dots, \\ \hat{X}_t(-1) = X_{t-1}, \hat{X}_t(0) = X_t. \end{cases}$$

由于特征方程

$$\lambda^2 - \frac{1}{25} = 0$$

有相异实根

$$\lambda_1 = \frac{1}{5}, \quad \lambda_2 = -\frac{1}{5},$$

因此

$$\hat{X}_t(l) = \frac{c_1}{5^l} + \frac{c_2}{(-5)^l}, \quad l = -1, 0, 1, \dots,$$

其中 c_1, c_2 由下式确定:

$$X_t = c_1 + c_2, \quad X_{t-1} = 5(c_1 - c_2),$$

即

$$c_1 = \frac{X_t}{2} + \frac{X_{t-1}}{10}, \quad c_2 = \frac{X_t}{2} - \frac{X_{t-1}}{10}.$$

因而

$$\begin{aligned}\hat{X}_t(l) &= \frac{1}{5^l} \left(\frac{X_t}{2} + \frac{X_{t-1}}{10} \right) + \frac{1}{(-5)^l} \left(\frac{X_t}{2} - \frac{X_{t-1}}{10} \right) \\ &= \frac{1}{2} \left(\frac{1}{5^l} + \frac{1}{(-5)^l} \right) X_t + \frac{1}{10} \left(\frac{1}{5^l} - \frac{1}{(-5)^l} \right) X_{t-1}, \quad l = -1, 0, 1, \dots\end{aligned}$$

$$\begin{aligned}e_t(l) &= X_{t+l} - \hat{X}_t(l) \\ &= X_{t+l} - \frac{1}{2} \left(\frac{1}{5^l} + \frac{1}{(-5)^l} \right) X_t - \frac{1}{10} \left(\frac{1}{5^l} - \frac{1}{(-5)^l} \right) X_{t-1}, \quad l = -1, 0, 1, \dots\end{aligned}$$

因而

$$\begin{aligned}E[e_t(l)^2] &= E \left\{ \left[X_{t+l} - \frac{1}{2} \left(\frac{1}{5^l} + \frac{1}{(-5)^l} \right) X_t - \frac{1}{10} \left(\frac{1}{5^l} - \frac{1}{(-5)^l} \right) X_{t-1} \right]^2 \right\} \\ &= E(X_{t+l}^2) + \left[\frac{1}{2} \left(\frac{1}{5^l} + \frac{1}{(-5)^l} \right) \right]^2 E(X_t^2) + \left[\frac{1}{10} \left(\frac{1}{5^l} - \frac{1}{(-5)^l} \right) \right]^2 E(X_{t-1}^2) \\ &\quad - \left(\frac{1}{5^l} + \frac{1}{(-5)^l} \right) E(X_{t+l}X_t) - \left(\frac{1}{5^{l+1}} + \frac{1}{(-5)^{l+1}} \right) E(X_{t+l}X_{t-1}) \\ &= \left\{ 1 + \left[\frac{1}{2} \left(\frac{1}{5^l} + \frac{1}{(-5)^l} \right) \right]^2 + \left[\frac{1}{10} \left(\frac{1}{5^l} - \frac{1}{(-5)^l} \right) \right]^2 \right\} \gamma(0) \\ &\quad - \left(\frac{1}{5^l} + \frac{1}{(-5)^l} \right) \gamma(l) - \left(\frac{1}{5^{l+1}} + \frac{1}{(-5)^{l+1}} \right) \gamma(l+1), \quad l = 1, 2, \dots\end{aligned}$$

其中自协方差函数如下给出:

$$\begin{cases} \gamma_j = 0 & \text{if } j = 2k+1, k \geq 0 \\ \gamma_j = \frac{(1/25)^k}{1-(1/25)^2} & \text{if } j = 2k, k \geq 0 \end{cases}$$